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Beyond the Formula: Why Pre-Service Teachers Stumble with Mathematical Induction in Word Problems

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This study examines the challenges pre-service teachers face in using mathematical induction to prove a complex word problem. Problem-solving, particularly with non-routine word problems, develops higher-order thinking skills, and proof, especially mathematical induction, is a core mathematical practice. For students and pre-service teachers, mathematical induction poses major challenges despite its importance. The present study identifies the technical, mathematical and conceptual difficulties that pre-service teachers encounter. A qualitative study was conducted with 34 pre-service teachers (aged 19–20). Data were collected through a written test (a non-routine mathematical induction word problem), questionnaires and semi-structured interviews. The analysis, guided by Avital and Libeskind's classification, revealed that technical difficulties include incorrect first case determination and failure to generalise patterns in the induction step. Mathematical difficulties stem from misinterpreting the problem and incorrectly defining the proposition $P(n)$. Conceptual difficulties involve misunderstanding the deductive nature of mathematical induction, rigid adherence to a first case of $n = 1$, and underestimating the importance of the basis step. The findings suggest that targeted pedagogical interventions in teacher training are required in order to improve pre-service teachers' modelling, conceptual grasp of mathematical induction and pattern-generalisation skills.

Keywords: mathematical induction, pre-service teachers, problem-solving, proof, word problems

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Onkraj formule: zakaj bodoči učitelji naletijo na težave pri matematični indukciji v besedilnih nalogah

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Ta študija preučuje izzive, s katerimi se bodoči učitelji spoprijemajo pri uporabi matematične indukcije za dokazovanje zapletenih besedilnih nalog. Reševanje problemov, zlasti z neobičajnimi besedilnimi nalogami, razvija veščine mišljenja višjega reda, dokazovanje, zlasti matematična indukcija, pa je osrednja matematična praksa. Za študente in bodoče učitelje matematična indukcija kljub svoji pomembnosti predstavlja velike izzive. Ta študija opredeljuje tehnične, matematične in konceptualne težave, s katerimi se spoprijemajo bodoči učitelji. Izvedli smo kvalitativno študijo s 34 bodočimi učitelji (starimi 19–20 let). Podatki so bili zbrani s pisnim testom (z neobičajno matematično indukcijsko besedilno nalogo), z vprašalniki in s polstrukturiranimi intervjuji. Analiza, ki je temeljila na klasifikaciji Avitala in Libeskinda, je pokazala, da tehnične težave vključujejo napačno določitev prvega primera in nezmožnost posplošitve vzorcev pri koraku indukcije. Matematične težave izhajajo iz napačne razlage problema in napačne opredelitve trditve $P(n)$. Konceptualne težave vključujejo napačno razumevanje deduktivne narave matematične indukcije, strogo upoštevanje prvega primera $n = 1$ in podcenjevanje pomena osnovnega koraka. Ugotovitve kažejo, da so za izboljšanje modeliranja, konceptualnega razumevanja matematične indukcije in veščin splošnega vzorca pri bodočih učiteljih potrebni ciljni pedagoški ukrepi v okviru usposabljanja učiteljev.

Ključne besede: matematična indukcija, bodoči učitelji, reševanje problemov, dokaz, besedilne naloge

Introduction

Problem-solving is a long-standing cornerstone of mathematics education (Liljedahl et al., 2016). This skill is crucial for linking theoretical knowledge with practical applications and for helping students navigate our constantly changing world (Csapó & Funke, 2017). Word problems are a crucial bridge, challenging students to apply mathematical concepts in a narrative context (Suseelan et al., 2023). Presenting students with non-routine word problems significantly enhances their problem-solving skills (Jäder et al., 2020), as these tasks trigger high cognitive demands (van Zanten & van den Heuvel-Panhuizen, 2018) and stimulate higher-order thinking skills (HOTS) (Kablan & Uğur, 2021).

In line with the emphasis on problem-solving, proof is also recognised as a core mathematical practice and an important foundation in mathematics education (Harel & Sowder, 1998; Stylianides et al., 2016). This has led to the view that proof should be central to every student's mathematical experience, even from the elementary level (NCTM, 2000; Schoenfeld, 1994). Mathematical induction (MI) is one of the most commonly used proving methods. Despite its importance, MI is widely recognised as challenging for learners at all levels, including pre-service teachers (PSTs) (Avital & Libeskind, 1978; Ernest, 1984; Fischbein & Engel, 1989; Movshovitz-Hadar, 1993; Stylianides et al., 2007).

When students learn MI, there is often a distinct gap between their experiences and teacher expectations (Norton et al., 2023). This difficulty can arise when students are encouraged to generate knowledge based on educator expectations rather than developing their own personal understanding (Dawkins & Weber, 2017). Papadopoulos and Kyriakopoulou (2022) found that reading mathematical texts, specifically MI proofs, can be viewed as a problem-solving activity, requiring students to pass through Polya's four phases. They highlighted the fact that the technical complexity and often unexplained knowledge assumptions in mathematical proofs make them difficult to read. This observation deepens our understanding of MI challenges, where the text itself becomes a major hurdle.

Previous research has highlighted various aspects of teachers' understanding of proof. For example, Knuth (2002a, 2002b) found that teachers often have a limited pedagogical view, seeing proof primarily as a subject of study rather than a means of teaching and learning mathematics. Moreover, many teachers hold restricted views on the nature of proof and display inadequate understanding of what constitutes a valid proof. This aligns with the findings of Stylianides et al., (2017), who emphasised the importance of teachers' understanding of proof for facilitating student learning. Although a great deal of research explores students'

MI difficulties and teachers' pedagogical perspectives on proof, further work is needed to better understand how PSTs, as future educators, navigate these challenges, particularly in the context of complex word problems.

Considering the fact that word problems, particularly those involving strong MI, can be a highly effective tool for high cognitive demands and stimulating HOTS, it is important to investigate how PSTs solve these types of problems. A better understanding of the difficulties they encounter will not only contribute to the proof education literature but also provide crucial insights for improving mathematics teacher training.

Mathematical Induction

Mathematical induction (MI) is an approach to proving propositions about the natural numbers. Its fundamental concepts are clear and appealing, while it has many applications and offers a consistent framework for learning natural numbers (Townsend, 1987). Like a game of dominoes designed so that the fall of the first domino triggers a series of subsequent falls, the principle of MI works by proving the truth of a basis step (the initial domino falls) and an induction step. In its weak form, the induction step requires only the assumption that the th statement is true (the th domino falls) to prove the th statement (the next domino falls). However, in strong MI, the induction step is "stronger": we assume that all statements before the th statement are true (all dominoes before a certain position have fallen) to prove the truth of the th statement. Although the induction hypotheses are different, these two forms of induction are actually equivalent and can be derived from each other.

The present study employs strong MI, defined by Townsend (1987) as follows: "Suppose that is some assertion about the natural number and that is some fixed natural number. To prove that is true for all it is sufficient to show the following: (1) that the first case is true; (2) that if and the cases are true, then the next case is true" (p. 5). Formal proof by MI has a procedural component, which refers to the skill to perform or understand proof by formal MI, and a conceptual component, which refers to a deeper understanding of proof techniques, including understanding why the basis step is essential, the role of the induction step, and why these two steps together allow the conclusion that the proposition is true for all natural numbers (Baker, 1996; Harel, 2001; Lowenthal & Eisenberg, 1992; Woodall, 1981).

Difficulties in Solving Word Problems

Word problems are combinations of numbers and words to which students apply mathematical instruction in the context of problem-solving

(Pfannenstiel et al., 2015; Wyndhamn & Säljö, 1997). Previous studies have revealed that students encounter at least three difficulties when solving mathematical word problems.

First, there is difficulty reading and understanding the presented circumstance, which is caused by a lack of the language and literacy skills (Pongsakdi et al., 2020) required to assist students in reading and understanding the problem requirement provided in a written description, as well as in interpreting it in a mathematical context (Reid O'Connor & Norton, 2022). Second, there is difficulty in constructing accurate mathematical sentences (Reid O'Connor & Norton, 2022), which stems from students' failure to recognise the variables presented in a question, understand the connections between those variables, and reveal the variable that must be obtained depending on the problem's requirements (Fuchs et al., 2020). Third, there is the difficulty of applying proper mathematical operations to variables, which is caused by a lack of conceptual knowledge, and results in students failing to comprehend the real purpose for using a specific mathematical operation (Crooks & Alibali, 2014)or knowledge of how to solve problems and enact procedures. In recent years, however, there has been a shift toward focusing, not only on solving problems, but also on conceptual knowledge. In the current work, we reviewed (1. This difficulty may also be caused by students' lack of procedural skills, which prevents them from using the arithmetic procedure to reach the solution (Daroczy et al., 2015).

Proving Word Problems Using MI

Numerous empirical studies have shown that students experience difficulties in logical reasoning (Markovits & Quinn, 2002), especially MI (Baker, 1996; Michaelson, 2008; G. J. Stylianides et al., 2007). Research by Baker (1996) showed that students experience difficulties both procedurally and conceptually, which are influenced by knowledge of mathematical content through examples. Research by Stylianides et al. (2007)showed that the difficulties experienced by preservice teachers centred on the essence of the basis step, the meaning of the induction step, and the use of mathematical content knowledge outside of MI. Avital and Libeskind (1978) classify the difficulties in MI into three categories: technical, mathematical and conceptual. Technical difficulties are related to students' difficulties in carrying out the steps needed to develop evidence, while mathematical difficulties concern the misinterpretation of the mathematical application of MI. Conceptual difficulties – such as failing to distinguish between deductive and inductive reasoning or underestimating the importance of the basis step – are particularly significant, since they resonate with de Villiers' (1990) argument that the role of proof extends beyond mere verification to include explanation, systematisation and discovery. These neglected functions are often overlooked by students and are at the heart of their conceptual challenges with MI.

Among the difficulties related to MI are those experienced if the problem is presented in the form of a word problem (Lubis & Nasution, 2017), which is the most difficult type of problem encountered by students (Verschaffel et al., 2020). Research by Lubis and Nasution (2017) showed that understanding the context of word problems plays an important role in how students solve logical reasoning problems. One group the mathematics problems that is considered difficult is questions about patterns of reasoning, such as questions that require the skill to determine a pattern (Kurniati et al., 2015).

On the background of the problems described above, the present study aims to describe PSTs' difficulties in proving problems in the form of word problems using MI. In order to achieve this goal, this study will answer the following research questions:

RQ1. What technical difficulties do PSTs experience in proving word problems using MI?
RQ2. What mathematical difficulties do PSTs experience in proving word problems using MI?
RQ3. What conceptual difficulties do PSTs experience in proving word problems using MI?

Method

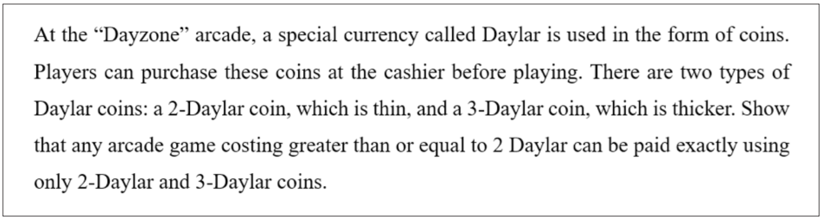
Particiants

The participants in the study were 34 pre-service teachers (PSTs) (5 males and 29 females) aged 19-20 years. They were enrolled in the Discrete Mathematics course in the third semester at a prestigious educational university in Indonesia. This group of participants was selected because they had initial exposure to the concept of mathematical induction (MI). The instructors in this course had covered a variety of examples of formal MI, including the first case that is not always and the concept of strong induction. The sample selection used a purposive sampling technique, focusing on PSTs who showed specific types of difficulties in proofs using MI based on initial analysis. This approach allowed the study to deeply explore the nuances of the difficulties experienced, in accordance with the descriptive purpose of the study.

Instruments

Data collection was carried out through three main instruments: written tests, questionnaires and interviews. The test sheet contains one MI word problem, as shown in Figure 1. This problem was chosen because it is relevant to the context of the Discrete Mathematics course and is designed as a non-routine word problem. This non-routine characteristic establishes high cognitive demands and effectively represents the higher-order thinking skills (HOTS) that are intended to be measured. The PSTs were instructed to explicitly indicate the patterns they used in the proof to solve this strong induction problem, which specifically has a first case other than . In mathematical terms, this problem is like a classic induction task (i.e., proving that every natural number greater than or equal to 2 can be expressed as a sum of multiples of 2 and 3), but it is presented in a more challenging verbal context. The test was administered in class under the lecturer's supervision; it lasted 50 minutes and was conducted after the completion of the mathematical induction material.

Figure 1
Mathematical induction test sheet



At the "Dayzone" arcade, a special currency called Daylar is used in the form of coins. Players can purchase these coins at the cashier before playing. There are two types of Daylar coins: a 2-Daylar coin, which is thin, and a 3-Daylar coin, which is thicker. Show that any arcade game costing greater than or equal to 2 Daylar can be paid exactly using only 2-Daylar and 3-Daylar coins.

A questionnaire was developed to complement the written test, consisting of five open-ended questions designed to systematically identify the different types of difficulties experienced by PSTs. This questionnaire serves as a supporting tool to validate and deepen understanding of the patterns of difficulties that emerge from the test.

The interviews were conducted in a semi-structured format, guided by five main questions to clarify the questionnaire responses, as well as to gain a deeper understanding of the PSTs' thought processes. Each interview, typically lasting 20–30 minutes, was recorded and transcribed. The main purposes of the interviews were: (1) to confirm that the PSTs understood the context of the word problem questions and that their initial interpretations did not diverge from the question's intent; (2) to confirm the results of the PSTs' written responses regarding the difficulties they experienced, thus allowing the researcher to clarify any ambiguities or gather additional details; and (3) to confirm the underlying causes of the difficulties the PSTs faced by investigating thoughts, strategies or misconceptions that might not be clear from written responses alone. This approach is crucial to understanding the roots of the problem causing MI proof difficulties.

Research Design

The results of the PSTs' work in solving problems and the results of filling out the questionnaire were classified based on the type of difficulty experienced in proving using MI. The classification of the difficulties used in the research refers to the categorisation of Avital and Libeskind (1978). Each difficulty was broken down into the sub-categories listed in Table 1. The main categories (technical, mathematical and conceptual) were applied deductively, while the sub-codes were developed inductively. Coding was carried out by two researchers serving as the primary independent raters, with the other researchers acting as reviewers, validators or arbiters in cases of disagreement. The inter-coder agreement exceeded 85%. Next, three PSTs representing each type of difficulty were selected to be interviewed. This selection was based on participants who showed variation or uniqueness in the types of difficulties found. The aim of the interview was to obtain complete information about the deep causes of the difficulties experienced, thus providing rich qualitative insights to support the quantitative analysis of the tests and questionnaires.

Table 1

Coding criteria and sample responses for the interview footage

Code	Sub-Code	Criteria
TD (technical difficulty)	RC (reading comprehension difficulty)	– Can't determine mathematical information known in the problem, or – can't determine what is asked in the problem.
	AR (arithmetical difficulty)	Performing wrong calculation.
	AL (algebraic difficulty)	Performing wrong algebraic operation.
	MM (mathematical modelling difficulty)	Can't translate the problem into proper mathematical language or representation.
MD (mathematical difficulty)	FC (first case difficulty)	– Can't determine the right first case, or – can determine the right first case in the mathematical model, but performing basis step with different case.
	BS (basis step difficulty)	– Can't perform the basis step, or – performing the basis step, but not adequate to find the pattern.
	FP (finding pattern difficulty)	– Can't find the patterns needed, or – can find some of the patterns needed, but not others.
	IS (induction step difficulty)	– Can't perform induction step, or – performing the induction step, but not adequate to draw a conclusion.

Code	Sub-Code	Criteria
CD (conceptual difficulty)	CC (certain cases difficulty)	– Assuming that the basis step always starts with a first case of $n = 1$, or – interpreting the induction step for $n = k + 1$ in certain cases.
	WI (wrong implication difficulty)	Drawing or making a wrong implication.

Results

Types of Pre-Service Teachers' Difficulties

Most of the 34 participating pre-service teachers (PSTs) demonstrated various types of difficulties in proving word problems using mathematical induction (MI), as presented in Table 2.

Table 2

Coding results for main categories

Code/Sub-code	Number of participants
TD (technical difficulty)	18 (52.9%)
– RC (reading comprehension)	4 (11.8%)
– AR (arithmetical)	2 (5.9%)
– AL (algebraic)	3 (8.8%)
– MM (mathematical modelling)	14 (41.2%)
MD (mathematical difficulty)	24 (70.6%)
– FC (first case)	9 (26.5%)
– BS (basis step)	10 (29.4%)
– FP (finding pattern)	12 (35.3%)
– IS (induction step)	16 (47.1%)
CD (conceptual difficulty)	5 (14.8%)
– CC (certain cases)	3 (8.9%)
– WI (wrong implication)	2 (5.9%)
No difficulties	3 (8.8%)

Further analysis revealed overlapping difficulties, as presented in Table 3.

Table 3

Coding results for overlapping main categories

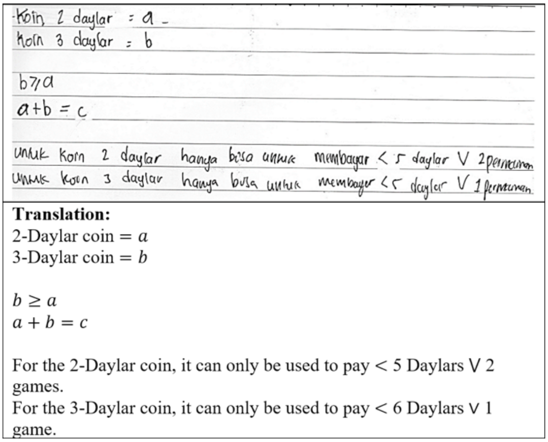
Code	Number of participants
TD and MD	10 (29.4%)
TD and CD	1 (2.9%)
MD and CD	3 (8.8%)
TD, MD and CD	1 (2.9%)

Causes of Pre-Service Teachers' Difficulties

In-depth analysis through interviews revealed the root causes of some of the identified difficulties.

S1: Mathematical Modelling and Information Interpretation Difficulties. S1 experienced a TD and MDs. His TD centred on the basis step, where he could not determine the correct first cases. His MDs related to misinterpreting the information in the problem and experiencing difficulty in determining the proposition $P(n)$. S1's answer is shown in Figure 2.

Figure 2
S1's answer



Koin 2 daylar = a
koin 3 daylar = b

$b \geq a$
$a+b=c$

untuk koin 2 daylar hanya bisa untuk membayar < 5 daylar V 2 permainan
untuk koin 3 daylar hanya bisa untuk membayar < 5 daylar V 1 permainan

Translation:
2-Daylar coin = a
3-Daylar coin = b

$b \geq a$
$a + b = c$

For the 2-Daylar coin, it can only be used to pay < 5 Daylars V 2 games.
For the 3-Daylar coin, it can only be used to pay < 6 Daylars ∨ 1 game.

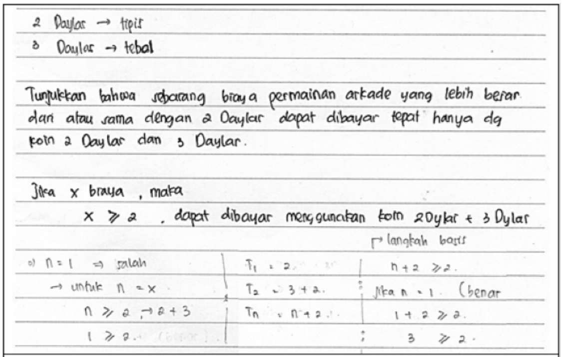
Below is an excerpt from the interview conducted with S1.

[...]
Researcher: What is your understanding of what was being asked in the problem?
S1: As far I know, the problem was asked to determine a cost greater than or equal to 2 Daylars. So, 2-Daylar or 3-Daylar coins can be used to play how many times.
Researcher: So as far you know, we have 2-Daylar and 3-Daylar coins?
S1: Yes. Suppose we have 2-Daylar and 3-Daylar coins, how many times can we play the game?
[...]

The S1 questionnaire responses and interview results indicated that S1's difficulties were caused by errors in identifying the problem or what was actually being asked. S1 interpreted the problem as asking how many games could be played, instead of demonstrating that every arcade fee of at least 2 Daylars can be paid exactly with 2-Daylar and 3-Daylar coins. This indicates a deficit in modelling the problem from a narrative context to a mathematical representation.

S2: Misconception of First Cases and Propositional Representation. S2 experienced a TD, an MD and a CD. Technically, he can determine the first cases but not precisely. Despite knowing that the proposition will be proven for *n* = ≥ 2, the process starts with different first cases. The MD is in determining *P*(*n*). S2's CD is the assumption that the basis step always starts with a first case of *n* = 1, regardless of the problem context. S2's answer is shown in Figure 3.

Figure 3
S2's Answer



2 Daylar → tipis
3 Daylar → tebal

Tunjukkan bahwa sebarang biaya permainan arkade yang lebih besar dari atau sama dengan 2 Daylar dapat dibayar tepat hanya dg koin 2 Daylar dan 3 Daylar.

Jika x biaya, maka
$x \geq 2$. dapat dibayar menggunakan koin 2Dylar + 3 Dylar

↱ langkah basis

a) $n = 1 \Rightarrow$ salah
→ untuk $n = x$
$n \geq 2 \rightarrow 2 + 3$
$1 \geq 2$ (benar)

$T_1 = 2$
$T_2 = 3 + 2$
$T_n = n + 2$

$n + 2 \geq 2$
Jika $n = 1$ (benar
$1 + 2 \geq 2$
$3 \geq 2$

Translation:
2 Daylars → thin
3 Daylars → thick

Show that any arcade game costing greater than or equal to 2 Daylars can be paid exactly with only 2-Daylars and 3-Daylars coins.

If x is the price, then
$x \geq 2$ can be paid using 2-Daylars + 3-Daylars coins.

$n = 1 \rightarrow$ false
$\rightarrow$ for $n = x$
$n \geq 2 \rightarrow 2 + 3$
$1 \geq 2$

$T_1 = 2$
$T_2 = 3 + 2$
$T_n = n + 2$

(Induction steps)
$n + 2 \geq 2$
If $n = 1$, (true)
$1 + 2 \geq 2$
$3 \geq 2$

Below is an excerpt from the interview conducted with S2.

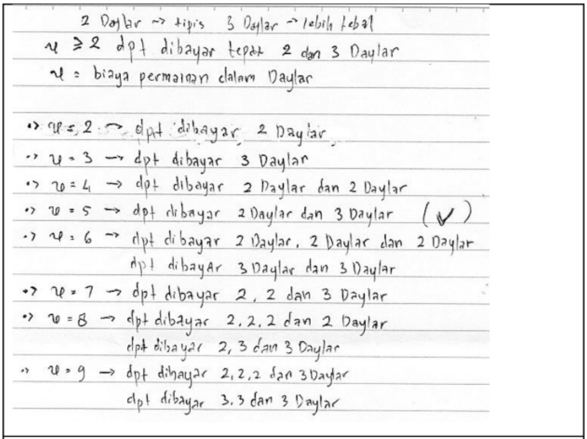
[...]
Researcher: What does x mean?
S2: I assumed that x is the number of coins we buy at Dayzone.
Researcher: Are n and x the same?
S2: Yes, they are the same.
Researcher: Why did you start the basis step with $n = 1$?
S2: Because in the basis step, we should assume that the first case is $n = 1$. Therefore, in my opinion, the n or x starts from one.
[...]
S2: Because $n = 1$ is wrong, I thought this couldn't be wrong. So, I did the basis step again by finding the recursive formula first.
Researcher: Where is this T_n from? $T_1 = 2$, $T_2 = 3 + 2$, then where did $T_n = n + 2$ come from?
S2: From the last sentence. I'm looking for a pattern; if T_1 is only 2 Daylars, T_2 is 3 Daylars plus 2 Daylars. Because the 2 Daylars will definitely count, so $T_n = n + 2$.
[...]

The S2 questionnaire responses and interview show that although S2 had an initial understanding of the need for a proof for $n \geq 2$, a deep misconception about first cases in induction hampered his progress. Failure to translate the problem into a proper mathematical representation, as seen by the incorrect attempt to find the recursive formula, exacerbated the difficulty in establishing $P(n)$ and the correct first case.

S3: Difficulty in Generalising Patterns in the Induction Step. S3 experienced MDs, especially in the induction step; namely, the difficulty in showing that $P(n + 1)$ is true if $P(n)$ is assumed to be true. Although S3 can verify specific cases for different values of x, he has difficulty in generalising. S3's answer is shown in Figure 4.

Figure 4

S3's Answer



2 Daylar → tipis 3 Daylar → lebih tebal

$x \geq 2$ dpt dibayar tepat 2 dan 3 Daylar

x = biaya permainan dalam Daylar

•> $x = 2$ → dpt dibayar 2 Daylar

•> $x = 3$ → dpt dibayar 3 Daylar

•> $x = 4$ → dpt dibayar 2 Daylar dan 2 Daylar

•> $x = 5$ → dpt dibayar 2 Daylar dan 3 Daylar (✓)

•> $x = 6$ → dpt dibayar 2 Daylar, 2 Daylar dan 2 Daylar
dpt dibayar 3 Daylar dan 3 Daylar

•> $x = 7$ → dpt dibayar 2, 2 dan 3 Daylar

•> $x = 8$ → dpt dibayar 2, 2, 2 dan 2 Daylar
dpt dibayar 2, 3 dan 3 Daylar

•> $x = 9$ → dpt dibayar 2, 2, 2 dan 3 Daylar
dpt dibayar 3, 3 dan 3 Daylar

Translation:

2 Daylar → thin 3 Daylar → thicker

$x \geq 2$ → can be paid exactly using 2-Daylar and 3-Daylar coins.

x = game cost in Daylars

$x = 2$ → can be paid with one 2-Daylar coin

$x = 3$ → can be paid with one 3-Daylar coin

$x = 4$ → can be paid with two 2-Daylar coins

$x = 5$ → can be paid with one 2-Daylar coin and one 3-Daylar coin (√)

$x = 6$ → can be paid with 2-Daylar + 2-Daylar + 2-Daylar
or with 3-Daylar + 3-Daylar

$x = 7$ → can be paid with 2-Daylar + 2-Daylar + 3-Daylar

$x = 8$ → can be paid with 2-Daylar + 2-Daylar + 2-Daylar + 2-Daylar
or with 2-Daylar + 3-Daylar + 3-Daylar

$x = 9$ → can be paid with 2-Daylar + 2-Daylar + 2-Daylar + 3-Daylar
or with 3-Daylar + 3-Daylar + 3-Daylar

Below is an excerpt from the interview conducted with S3.

[...]

Researcher: Suppose I take a random number, $x = 15$, can it be paid with 2-Daylar and 3-Daylar coins?

S3: Yes, it takes three 2-Daylar coins and three 3-Daylar coins.

Researcher: How about $x = 16$?

S3: It takes two 2-Daylar coins and four 3-Daylar coins.

Researcher: Well, you can determine the number of coins for a random number. Now, what guarantee is there that if you can show that it is true for $x = k$, then it must be true for $x = k + 1$ as well?

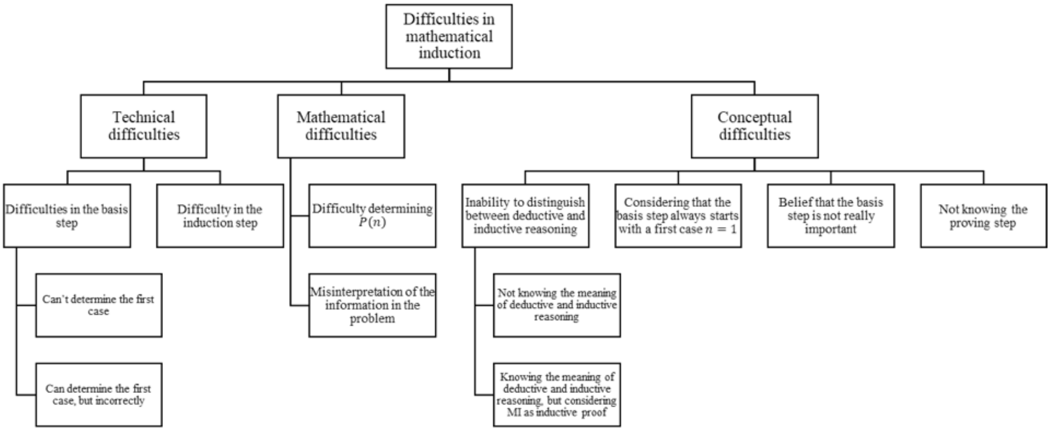
S3: Look for the pattern.

Researcher: What is the pattern?
S3: That one is still confused, making the induction step is still confusing.
The S3 questionnaire responses and the interview with S3 highlight the difficulty in generalising from specific cases to general patterns, which is the essence of the induction step. Although S3 understood the need to find patterns, his inability to formulate them mathematically prevented him from completing the induction step.

Discussion

Based on the results described above, various types of difficulties in mathematical induction (MI) were identified among the participating pre-service teachers (PSTs), as presented in Figure 5. The results suggest that these difficulties are not isolated, but can occur simultaneously and even influence each other. In particular, technical difficulties (TDs) and mathematical difficulties (MDs) were the most common types of difficulties and usually occurred together, whereas conceptual difficulties (CDs) were less common but had a big impact on the PSTs' fundamental understanding. These findings are consistent with the classical categorisation of difficulties by Avital and Libeskind (1978), but our data provide a nuanced perspective of the prevalence and interconnections between various types of difficulties within the context of PSTs in Indonesia.

Figure 5
Types of difficulty in mathematical induction



The MDs experienced by the PSTs are mostly due to their inability to identify the problem or specific terms in the word problem, as well as their inability to translate the problem into the right mathematical language or representation. This is clearly seen in the case of S1, who misinterpreted the core of the problem question. The text comprehension model of Kintsch and van Dijk (1978) emphasises the fact that an important step in solving word problems is constructing a mental representation of the real problem, called a situational model, and transforming the situational model into a mathematical model. Previous research by Leiss et al. (2019a) consistently shows that constructing a situational model is crucial for determining the correct solution, and that this process generally takes around 40% of the total processing time, depending on the language complexity of the word problems. The accuracy of the situational models produced and the students' success in developing mathematical models are both significantly affected by their reading comprehension skills, both generally and in particular domains (Leiss, 2010; Leiss et al., 2019). Our findings provide compelling evidence for this argument, showing how deficiencies in early comprehension and modelling directly hinder PSTs' skill to properly initiate MI evidence.

Meanwhile, the TDs experienced by the PSTs in the basis steps are generally closely related to the underlying MDs. When the PSTs fail to translate the problem correctly, they will inherently have difficulty in determining the correct first case or formulating the proposition correctly, as seen in the case of S2. Furthermore, TDs in the induction step often arise from the PSTs inability to find and generalise patterns from the basis steps that have been carried out, as experienced by S3. This indicates a lack of one of the heuristic or experience-based techniques needed in problem-solving, which is the skill to organise data in a systematic list and look for patterns to identify the relationship between the data and the problem or what is being asked in the problem (Mousoulides & Sriraman, 2014). Previous research by Pedemonte (2007) also emphasised that the construction of MI is highly dependent on the type of generalisation used in the induction step, whereby students can construct MI only if the generalisation is based on the process (Harel, 2001). Our findings suggest that although PSTs may be able to verify specific cases, the real challenge occurs when they need to abstract patterns and generalise them into algebraic expressions that can be used in the induction step.

Although less common in our sample, CDs indicate a deep misunderstanding of the nature of MI. One form of CD is the inability to distinguish between deductive and inductive reasoning, which is caused by PSTs' assumption that MI involves generalising rules from observations of a limited number of cases (in the basis step). Despite its name, formal MI is actually a form of deductive reasoning; based on the axiom of natural numbers, it is important to show that the results hold for all natural numbers (Relaford-Doyle & Núñez, 2021). Meanwhile, the pattern generalisation process in inductive argumentation functions as a bridge between inductive argumentation and deductive proof (Martinez & Pedemonte, 2014).

Another form of CD is the consideration that the first case always starts with the first case $n = 1$, which is caused by PSTs' assumption that MI must apply to all natural numbers, which start at 1. It is a common misconception that that the case with $n = 1$ must always be present in the basis step (Stylianides et al., 2007). This misconception is also exacerbated by the tendency of students to memorise the MI structure and perform the proof mechanically (Pang & Dindyal, 2012; Ron & Dreyfus, 2004). In fact, a study by Larson and Pettersson (2018) showed that this misconception occurs even when the first case is given in the problem. Our study supports this observation: despite knowing that the proposition applies to $n \geq 2$, S2 still tried to start with $n = 1$ due to this misperception, demonstrating its ingrained nature.

Lastly, CDs in the form of the belief that the basis step is not really important are caused by the experience of PSTs who always find that the proposition applies to a first case. Previous research has shown that some students believe that the basis step must be verified before the induction step in order for the proof to be valid (Pang & Dindyal, 2012), while other students believe that the basis step can always be verified, and thus only the induction step is important (Stylianides et al., 2007). This results in the verification of the basis step often being treated as a meaningless formality and not considered as essential to the proof (Dubinsky, 1986; Ernest, 1984; Palla et al., 2012); alternatively, it is regarded as a first step to give assurance that the assertion to be proved is true (Ron & Dreyfus, 2004). It is important to note that the basis step is not a redundant supposition; even if the induction step is accurate, the propositional function is not true for all natural numbers if the basis step is false (García-Martínez & Parraguez, 2017).

The implications of our findings highlight the need to move beyond deficit-based perspectives in understanding PSTs' engagement with mathematical induction. While the study documents a variety of difficulties, it also reveals that PSTs possess a foundation of knowledge and reasoning skills that can be productively harnessed. Recognising these resources shifts the focus from cataloguing errors to examining how existing competencies – such as verifying specific cases or experimenting with strategies – can serve as entry points for more formal reasoning. This contribution emphasises a reconceptualisation of how we view PSTs: not merely as learners struggling with abstract proof, but as individuals whose partial understandings and emerging strategies provide valuable footholds for instructional design.

Conclusion

The present study concludes that pre-service teachers (PSTs) experience various difficulties in mathematical induction (MI), which can be categorised into three main types: technical difficulties (TDs), mathematical difficulties (MDs) and conceptual difficulties (CDs).

TDs mainly occur at two key stages of MI proofs: the basis step and the induction step. In the basis step, PSTs have difficulty in determining the correct first cases, and even if they can determine them, they often do so incorrectly. Meanwhile, in the induction step, TDs are dominated by the inability to show that $P(n + 1)$ is true if $P(n)$ is assumed to be true, often due to the inability to generalise the required pattern.

MDs are rooted in the understanding and interpretation of the problem. PSTs face difficulties in determining the proposition $P(n)$ correctly and often misinterpret the information given in the word problem. This indicates a fundamental challenge in transforming verbal problems into accurate mathematical representations.

Finally, CDs reveal a deeper misunderstanding of the nature of MI. These difficulties include the inability to distinguish between deductive and inductive reasoning, even when PSTs know the definitions of both but still consider MI to be inductive proof. In addition, PSTs often assume that the basis step always starts with a first case of $n = 1$, regardless of the problem context. Some PSTs also tend to believe that the basis step is not very important in the structure of the proof, while others do not understand the steps of the MI proof as a whole. These conceptual misunderstandings significantly hamper the ability of PSTs to construct valid and logical inductive arguments.

Given the complexity of the difficulties encountered, further research is needed to explore coping strategies and to determine how to effectively address each type of difficulty. Future studies could focus on developing and testing pedagogical interventions designed to: (1) enhance PSTs' mathematical modelling skills, particularly in translating word problems into correct mathematical notation and propositions; (2) strengthen PSTs' conceptual understanding of the deductive nature of MI and the importance of each step of the proof, including the significance of the basis step and the flexibility of determining the first case; and (3) develop PSTs' skills to identify and generalise patterns as a foundation for the induction step.

Future research could adopt an asset-based approach (Celedón-Pattichis et al., 2018) in order to identify PSTs' existing strengths and knowledge, so that interventions can build on a solid foundation rather than focusing only on weaknesses. Focusing on "what can be done" by PSTs will provide a more holistic and constructive perspective in designing MI curricula and teaching methods.

The present study is not without its limitations. The relatively small sample size means that the findings cannot be generalised too broadly, and the focus on a single type of non-routine MI problem may not reflect the full range of challenges that pre-service teachers encounter. In addition, because part of the data came from self-reported interviews, there is the possibility of subjective bias.

Even so, the study offers several valuable practical insights. It points to the importance of teacher education programmes that include carefully designed problem sets encouraging flexible approaches to the basis step, structured opportunities to practise modelling and generalising patterns, and explicit discussions about the deductive nature of MI. Integrating such approaches into mathematics instruction could help pre-service teachers not only improve their problem-solving skills and conceptual understanding of proof, but also develop stronger tools for supporting their future students' learning.

Ethical Statement

Ethical approval for this study was obtained from Universitas Negeri Malang. All of the participants provided informed consent prior to their involvement in the study, and their anonymity and confidentiality were strictly maintained throughout the research process. The research was conducted in accordance with the ethical guidelines and regulations of Universitas Negeri Malang.

Data Availability Statement

The data supporting the findings of this study are not publicly available due to the presence of student personal data but are available from the corresponding author upon reasonable request.

Disclosure Statement

On behalf of all authors, the corresponding author reports no potential conflict of interest.

No generative AI tools were used in the preparation or writing of this manuscript and the authors accept full responsibility for the content and integrity of the publication.

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References

Avital, S., & Libeskind, S. (1978). Mathematical induction in the classroom: Didactical and mathematical issues. *Educational Studies in Mathematics*, *9*(4), 429–438. JSTOR.

Baker, J. D. (1996, April 8–12). *Students' difficulties with proof by mathematical induction* [Paper presentation]. Annual Meeting of the American Educational Research Association, New York City, NY, United States.

Celedón-Pattichis, S., Borden, L. L., Pape, S. J., Clements, D. H., Peters, S. A., Males, J. R., Chapman, O., & Leonard, J. (2018). Asset-Based approaches to equitable mathematics education research and practice. *Journal for Research in Mathematics Education JRME*, *49*(4), 373–389. https://doi.org/10.5951/jresematheduc.49.4.0373

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review*, *34*(4), 344–377. https://doi.org/10.1016/j.dr.2014.10.001

Csapó, B., & Funke, J. (2017). *The nature of problem solving: Using research to inspire 21st century learning*. OECD Publishing. https://doi.org/10.1201/9781003160618-1

Daroczy, G., Wolska, M., Meurers, W. D., & Nuerk, H.-C. (2015). Word problems: A review of linguistic and numerical factors contributing to their difficulty. *Frontiers in Psychology*, *6,* Article 348. https://www.frontiersin.org/journals/psychology/articles/10.3389/fpsyg.2015.00348

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123–142. https://doi.org/10.1007/s10649-016-9740-5

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17–24.

Dubinsky, E. (1986). Teaching mathematical induction: I. *The Journal of Mathematical Behavior*, *5*(3), 305–317.

Ernest, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics*, *15*(2), 173–189. https://doi.org/10.1007/BF00305895

Fischbein, E., & Engel, I. (1989). Psychological difficulties in understanding the principle of mathematical induction. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th international conference for the psychology of mathematics education Vol I* (pp. 276–282). CNRS.

Fuchs, L., Fuchs, D., Seethaler, P. M., & Barnes, M. A. (2020). Addressing the role of working memory in mathematical word-problem solving when designing intervention for struggling learners. *ZDM*, *52*(1), 87–96. https://doi.org/10.1007/s11858-019-01070-8

García-Martínez, I., & Parraguez, M. (2017). The basis step in the construction of the principle of mathematical induction based on APOS theory. *The Journal of Mathematical Behavior*, *46*, 128–143. https://doi.org/10.1016/j.jmathb.2017.04.001

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. *Journal of Mathematical Behavior*, *20*(2), 185–212. https://mathweb.ucsd.edu/~harel/The%20Development%20of%20Mathematical%20Induction%20as%20a%20Proof%20Scheme%20-%20A%20Model%20for%20DNR-Based%20Instruction.pdf

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Issues in mathematics education: Vol. 7. Research in collegiate mathematics education III* (pp. 234–283). American Mathematical Society.

Jäder, J., Lithner, J., & Sidenvall, J. (2020). Mathematical problem solving in textbooks from twelve countries. *International Journal of Mathematical Education in Science and Technology*, *51*(7), 1120–1136. https://doi.org/10.1080/0020739X.2019.1656826

Kablan, Z., & Uğur, S. S. (2021). The relationship between routine and non-routine problem solving and learning styles. *Educational Studies*, *47*(3), 328–343. https://doi.org/10.1080/03055698.2019.1701993

Kintsch, W., & van Dijk, T. A. (1978). Toward a model of text comprehension and production. *Psychological Review*, *85*(5), 363–394. https://doi.org/10.1037/0033-295X.85.5.363

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education JRME*, *33*(5), 379–405. https://doi.org/10.2307/4149959

Knuth, E. J. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61–88. https://doi.org/10.1023/A:1013838713648

Kurniati, K., Kusumah, Y. S., Sabandar, J., & Herman, T. (2015). Mathematical critical thinking ability through. *Journal on Mathematics Education*, *6*(1), 53–62.

Larson, N., & Pettersson, K. (2018). Proof by induction-the role of the induction basis. In E. Noren, H. Palmer, & A. Cooke (Eds.), *Nordic research in mathematics education: papers of NORMA 17* (pp. 99–107). SMDF.

Leiss, D. (2010). Adaptive lehrerinterventionen beim mathematischen modellieren – Empirische befunde einer vergleichenden labor- und unterrichtsstudie [Adaptive teacher interventions in mathematical modelling – Empirical findings of a comparative laboratory and classroom study]. *Journal Für Mathematik-Didaktik*, *31*(2), 197–226. https://doi.org/10.1007/s13138-010-0013-z

Leiss, D., Plath, J., & Schwippert, K. (2019). Language and mathematics—Key factors influencing the comprehension process in reality-based tasks. *Mathematical Thinking and Learning*, *21*(2), 131–153. https://doi.org/10.1080/10986065.2019.1570835

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem solving in mathematics education*. Springer Nature. https://doi.org/10.1007/978-94-007-4978-8_129

Lowenthal, F., & Eisenberg, T. (1992). Mathematical induction in school: An illusion of rigor? *School Science and Mathematics*, *92*(5), 233–238. https://doi.org/10.1111/j.1949-8594.1992.tb15580.x

Lubis, A., & Nasution, A. A. (2017). How do higher-education students use their initial understanding to deal with contextual logic-based problems in discrete mathematics? *International Education Studies*, *10*(5), 72–86. https://doi.org/10.5539/ies.v10n5p72

Markovits, H., & Quinn, S. (2002). Efficiency of retrieval correlates with "logical" reasoning from causal conditional premises. *Memory & Cognition*, *30*(5), 696–706. https://doi.org/10.3758/BF03196426

Martinez, M. V., & Pedemonte, B. (2014). Relationship between inductive arithmetic argumentation and deductive algebraic proof. *Educational Studies in Mathematics*, *86*(1), 125–149. https://doi.org/10.1007/s10649-013-9530-2

Michaelson, M. T. (2008). A literature review of pedagogical research on mathematical induction. *Australian Senior Mathematics Journal*, *22*(2), 57–62.

Mousoulides, N., & Sriraman, B. (2014). Heuristics in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 253–255). Springer. https://doi.org/10.1007/978-94-007-4978-8

Movshovitz-Hadar, N. (1993). The false coin problem, mathematical induction and knowledge fragility. *The Journal of Mathematical Behavior*, *12*(3), 253–268.

NCTM. (2000). *Principles and standards for school mathematics*. NCTM Publisher.

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2023). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295–321. https://doi.org/10.1007/s40753-022-00163-2

Palla, M., Potari, D., & Spyrou, P. (2012). Secondary school students' understanding of mathematical induction: Structural characteristic and the process of proof construction. *International Journal of Science and Mathematics Education*, *10*(5), 1023–1045.

Pang, A. W.-K., & Dindyal, J. (2012). Students' reasoning errors in writing proof by mathematical induction. In B. Kaur, & T. L. Toh (Eds.), *Reasoning, communication and connections in mathematics* (Vol. 1–0, pp. 215–237). World Scientific. https://doi.org/10.1142/9789814405430_0011

Papadopoulos, I., & Kyriakopoulou, P. (2022). Reading mathematical texts as a problem-solving activity: The case of the principle of mathematical induction. *Center for Educational Policy Studies Journal*, *12*(1), 35–53. https://doi.org/10.26529/cepsj.881

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41. https://doi.org/10.1007/s10649-006-9057-x

Pfannenstiel, K. H., Bryant, D. P., Bryant, B. R., & Porterfield, J. A. (2015). Cognitive strategy instruction for teaching word problems to primary-level struggling students. *Intervention in School and Clinic*, *50*(5), 291–296. https://doi.org/10.1177/1053451214560890

Pongsakdi, N., Kajamies, A., Veermans, K., Lertola, K., Vauras, M., & Lehtinen, E. (2020). What makes mathematical word problem solving challenging? Exploring the roles of word problem characteristics, text comprehension, and arithmetic skills. *ZDM*, *52*(1), 33–44. https://doi.org/10.1007/s11858-019-01118-9

Reid O'Connor, B., & Norton, S. (2022). Supporting indigenous primary students' success in problem-solving: Learning from Newman interviews. *Mathematics Education Research Journal*, *34*(2), 293–316. https://doi.org/10.1007/s13394-020-00345-8

Relaford-Doyle, J., & Núñez, R. (2021). Characterizing students' conceptual difficulties with mathematical induction using visual proofs. *International Journal of Research in Undergraduate Mathematics Education*, *7*(1), 1–20. https://doi.org/10.1007/s40753-020-00119-4

Ron, G., & Dreyfus, T. (2004). The use of models in teaching proof by mathematical induction. In M. J. Hoines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, Issue 1978, pp. 113–120). Bergen University College.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*(1), 55–80. https://doi.org/10.1016/0732-3123(94)90035-3

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutierez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 315–351). Sense Publishers.

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*(3), 145–166. https://doi.org/10.1007/s10857-007-9034-z

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). National Council of Teachers of Mathematics.

Suseelan, M., Chew, C. M., & Chin, H. (2023). School-type difference among rural grade four Malaysian students' performance in solving mathematics word problems involving higher order thinking skills. *International Journal of Science and Mathematics Education*, *21*(1), 49–69. https://doi.org/10.1007/s10763-021-10245-3

Townsend, M. (1987). *Discrete mathematics: Applied combinatorics and graph theory*. The Benjamin/Cummings Publishing Company, Inc.

van Zanten, M., & van den Heuvel-Panhuizen, M. (2018). Opportunity to learn problem solving in Dutch primary school mathematics textbooks. *ZDM*, *50*(5), 827–838. https://doi.org/10.1007/s11858-018-0973-x

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *ZDM*, *52*(1), 1–16. https://doi.org/10.1007/s11858-020-01130-4

Woodall, D. R. (1981). Finite sums, matrices and induction. *The Mathematical Gazette*, *65*(432), 92–103. Cambridge Core. https://doi.org/10.2307/3615728

Wyndhamn, J., & Säljö, R. (1997). Word problems and mathematical reasoning—A study of children's mastery of reference and meaning in textual realities. *Learning and Instruction*, *7*(4), 361–382. https://doi.org/10.1016/S0959-4752(97)00009-1

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